Probing Anderson localization of light via decay rate statistics

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We have studied the distribution of resonance widths $P(\Gamma)$ in one-, two-, and three-dimensional multiple light scattering systems. $P(\Gamma)$ should follow a universal power law $P(\Gamma)\sim\Gamma^{-1}$ in the localized regime as confirmed by extensive numerical calculations. This behavior can be interpreted as an unambiguous signature of exponential Anderson localization of light in open systems.

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The research on Anderson localization of light has been of great interest [1] since it was originally proposed as the optical counterpart of electronic localization [2]. Localization, as proposed by Anderson, is defined as an inhibition of wave diffusion in infinite disordered media due to interference of multiple scattered waves [3]. A much stronger definition is that the eigenfunctions in an infinite disordered medium are characterized by an exponential decay in space, $|\psi(\mathbf{r})|\sim\exp(-|\mathbf{r}-\mathbf{r}'|/\xi)$, where ξ is the localization length. In finite, *open* media, waves can "leak" through the sample boundaries. Anderson localization must thus relate to manifestations of leakage in observables quantities. For optical systems, they are typically the emerging intensity, the total transmission, or the coherent backscattering cone. The observation of an exponential scaling of transmission [4,5], as well as the rounding of the backscattering cone [6], may not have definitively established localization since absorption could be responsible for these same effects.

There are several criteria to determine the onset of the localized regime. The Ioffe-Regel criterion states that, in three dimensions (3D), localization occurs for $k\ell\sim 1$ (with k the light wave number inside the medium and ℓ the mean free path). Another approach observes electromagnetic localization from the variance of fluctuations of transmission, even in the presence of absorption [7]. In open systems, the "eigenstates" are resonances with a finite energy width Γ (or, equivalently, with a finite lifetime $t\sim 1/\Gamma$) due to leakage. The Thouless criterion asserts that localization can be said to occur when the typical time that an excitation needs to propagate through the entire system of size R, $t_T\sim 1/\Gamma_T\sim R^2/D$ (Thouless time), exceeds the maximal time scale of the system, $t_H\sim 1/\Gamma_H\sim 1/\Delta E$ (Heisenberg time) [8]. Here D is the diffusion constant and ΔE the mean level spacing.

This Thouless criterion applies to the *average* leakage width. Hence it is reasonable to assume that the *statistical properties* of resonance widths are strongly affected by localization. The aim of the present paper is to investigate how localization manifests itself in the distribution of resonance widths $P(\Gamma)$ in multiple light scattering in open systems. We will show that $P(\Gamma)$ exhibits the universal power law $P(\Gamma)\sim\Gamma^{-1}$ in 1D, 2D, and 3D optical disordered systems, thereby generalizing recent theoretical [9] and numerical [10] studies in 1D models of mesoscopic transport. We assert that the algebraic decay $P(\Gamma)\sim\Gamma^{-1}$ represents a universal property of Anderson localization of light in open systems of any dimension.

Although the statistical properties of resonance widths in open systems have been extensively studied over the last years, in particular for chaotic/ballistic systems [11–13], their behavior for disordered systems exhibiting localization has received considerably less attention. As argued by Casati *et al.*, $P(\Gamma)$ should follow a power law $P(\Gamma)\sim\Gamma^{-1}$ in localized, classically chaotic systems [14]. $P(\Gamma)$ was analytically obtained for 1D disordered systems, showing a slightly different power law $P(\Gamma)\sim\Gamma^{-1.25}$ [9]. This prediction was corroborated later by numerical calculations in 1D and quasi-1D tight binding models [10]. The $P(\Gamma)\sim\Gamma^{-1}$ behavior was also reported in 1D and 3D strongly driven atomic Rydberg states in the context of dynamical localization [15]. Exactly at the Anderson transition [16] and in the diffusive regime [17], $P(\Gamma)$ was shown to follow a power law with a power different from -1. Concerning the study of $P(\Gamma)$ for optical systems, the only work on the subject is, to the best of our knowledge, due to Patra [18], who mainly focused on the small Γ regime and its application to random lasers. For small Γ (i.e., for $\Gamma\lesssim\langle\Gamma\rangle$), it is known that $P(\Gamma)$ is different from a power law, for both the diffusive [12,18] and the localized [13,18] regimes. It should be emphasized that the power law decay of $P(\Gamma)$ is expected to occur only for $\Gamma\gtrsim\langle\Gamma\rangle$, with typically $\langle\Gamma\rangle\sim\Gamma_T$ in the diffusive regime and $\langle\Gamma\rangle\sim\exp(-R/\xi)$ in the localized regime. However, for very large Γ ($\Gamma\gg\langle\Gamma\rangle$) the resonances are strongly coupled to the continuum and $P(\Gamma)$ decays faster than algebraically, both in the diffusive [17] and in the localized [9,10] regimes.

We will present a simple physical argument, inspired by Refs. [9,10,14], to explain the *universal* $P(\Gamma)\sim\Gamma^{-1}$ behavior for the localized regime, i.e., independent of the dimensionality of the system. Due to the opening of the system, exponentially localized eigenstates of the corresponding closed system (linear size R) acquire a finite frequency width Γ', $\Gamma'\sim e^{-2r'/\xi}$, with r' the distance to the boundaries. Near the system boundaries, the leakage is strong and the resonances are broad compared to Γ_T. On the other hand, far

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from the boundaries the leakage is small and the typical Γ in this region is much smaller than Γ_T. Assuming that the resonances are—like the scatterers—uniformly distributed in space, it follows that the (integrated) probability of finding a resonance width Γ smaller than Γ', $P_{int}(\Gamma<\Gamma')$, is equal to the probability of finding a resonance situated at a distance r from the boundaries larger than r', $P(r>r')$, i.e., $P_{int}(\Gamma<\Gamma')=P(r>r')$. Since $P(r>r')\propto\mu_d(R-r')/\mu_d(R)$ with μ_d the d-dimensional volume, we conclude that the probability density is

$$P(\Gamma')=\frac{dr'}{d\Gamma'}\frac{d}{dr'}[P(r>r')]\propto-\frac{\xi}{\Gamma'}\frac{d}{dr'}\left[\frac{\mu_d(R-r')}{\mu_d(R)}\right]. \tag{1}$$

The purely geometrical factor $d/dr'[\mu_d(R-r')/\mu_d(R)]$ depends on the dimensionality of the system but does not affect the exponent in Γ.

To test the validity of Eq. (1) for Anderson localization of light, we will consider scalar wave propagation in disordered media using the method introduced by Rusek and Orlowski [19,20]. This approach is based on the analysis of the spectrum of the Green matrix, which describes light scattering from randomly distributed pointlike dipoles (i.e., particles much smaller than the wavelength of light). For an incident plane wave $\psi_0(\mathbf{r})$ in a system of N identical dipoles with scattering matrix t, the field acting in the dipole at $\mathbf{r}_i$ is given by [19,20]:

$$\psi(\mathbf{r}_i)=\psi_0(\mathbf{r}_i)+t\sum_{j\neq i}^{N}G(\mathbf{r}_{ij})\psi(\mathbf{r}_j). \tag{2}$$

The complex-valued $N\times N$ matrix $G(\mathbf{r}_{ij})$ describes light propagation of the wave scattered by the dipole at $\mathbf{r}_i$ to the dipole at $\mathbf{r}_j$. Since the eigenvalues λ_M of $\mathbf{M}\equiv\mathbf{I}-t\mathbf{G}$ and λ_G of $\mathbf{G}$ are related by $\lambda_M=1-t\lambda_G$, and t depends on frequency ω via the scattering phase shift $\delta(\omega)$ [21], an eigenvalue λ_G with $\mathrm{Re}\lambda_G=-1$ will facilitate an appropriate choice for $\delta(\omega)$ such that $\lambda_M=0$. This would correspond to a genuinely localized state somewhere inside the random medium [19]. Assuming a Breit-Wigner model for the scatterers (with one sharp resonance of width Γ_0 at the position ω_0), for which $\delta(\omega)$ has a simple form, it is possible to obtain, in a good approximation, the resonance widths Γ via λ_G, $\Gamma/\Gamma_0\simeq1+\mathrm{Re}\lambda_G$ [20]. We will numerically diagonalize $\mathbf{G}$ in 1D, 2D, and 3D and calculate the distribution of resonance widths $P(\Gamma)$ using the above approximation.

It is interesting to compare the typical values of the resonance widths Γ to the Thouless frequency Γ_T. To estimate Γ_T let us recall that $\Gamma_T=1/t_T=2dD_B/R^2$, where D_B is the Boltzmann diffusion constant and d the dimension of the system. The Boltzmann diffusion constant is given by $D_B=v_E\ell^*/d$, where ℓ^* is the transport mean free path (which is, for point scatterers, equal to ℓ) and v_E the energy transport velocity, $v_E\approx c_0/[1+\tau_{dwell}/\tau_{mf}]$ [22], with $\tau_{dwell}=1/\Gamma_0$ the dwell time in a single scattering and $\tau_{mf}=\ell/c_0$ the mean free time. The mean free path is given by $\ell=1/n\sigma_d$, with n the density of scatterers and σ_d the $(d-1)$-dimensional cross section of a single point scatterer. Applying these considerations, Γ_T can be written as $\Gamma_T/\Gamma_0\sim2(\ell/R)^2$. In what follows, we will always compare the values of Γ to Γ_T.

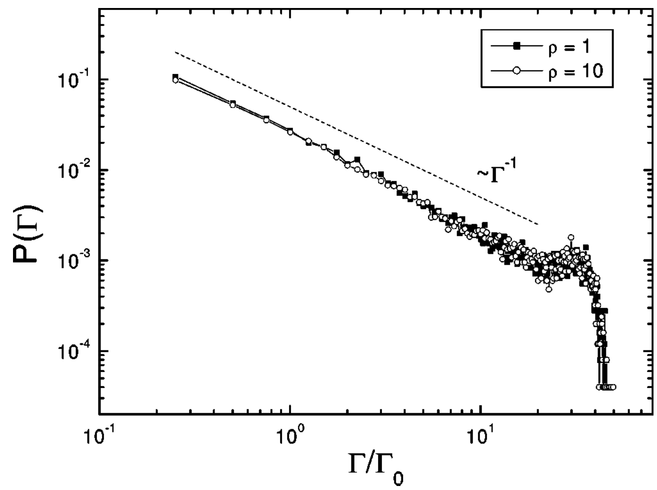

FIG. 1. The normalized distribution of resonance widths $P(\Gamma)$ calculated for 1000 different configurations of 100 point scatterers randomly distributed in a 1D segment with two different values of the uniform optical density ρ, $\rho=1$ (full squares) and $\rho=10$ (open circles) scatterers per wavelength. The dashed line corresponds to the prediction $P(\Gamma)\sim\Gamma^{-1}$ for the localized regime and the solid lines are just to guide the eyes. The values of Γ are normalized by the resonance width of a single dipole Γ_0. The value of the Thouless frequency for resonant scatterers is $\Gamma_T/\Gamma_0=2/N^2\simeq0.0002$.

In Fig. 1, $P(\Gamma)$ is calculated for 1D systems composed of 100 randomly distributed scatterers in a linear segment for two different values of the uniform optical density ρ: $\rho=1$ and $\rho=10$ scatterers per wavelength. In 1D, all eigenstates are known to be exponentially localized even for weak disorder and ξ is of the order of the mean free path ℓ. $P(\Gamma)$ is seen to exhibit a power law with an exponent very close to -1, in good agreement with Eq. (1). In addition, the exponent does not change by increasing ρ, i.e., by decreasing ξ. This demonstrates that the algebraic decay $P(\Gamma)\sim\Gamma^{-1}$ in the localized regime is valid not only for 1D models of mesoscopic transport [9,10], but also for our model of wave propagation in disordered media. At large Γ, $P(\Gamma)$ decays faster than algebraically. This can be explained by the fact that this region is dominated by short living resonances, typically close to the boundaries, for which the prediction (1) breaks down. To compare the values of Γ to the Thouless frequency Γ_T, let us recall that the (dimensionless) cross section of a point scatterer in 1D is simply the reflection coefficient. This implies that on resonance, $\Gamma_T/\Gamma_0\sim2(\ell/R)^2=2/N^2$ with N the number of scatterers. For $N=100$ as in Fig. 1, we have $\Gamma_T/\Gamma_0\simeq0.0002$, showing that the values of Γ in Fig. 1 are far above the Thouless frequency Γ_T. The Thouless frequency Γ_T does not represent the appropriate characteristic decay rate for 1D systems since diffusion never occurs.

Figure 2 shows $P(\Gamma)$ for 2D systems containing $N=2500$ scatterers randomly distributed in a $R\times R$ square for $\rho=1$ and $\rho=10$ scatterers per wavelength squared. In 2D, in

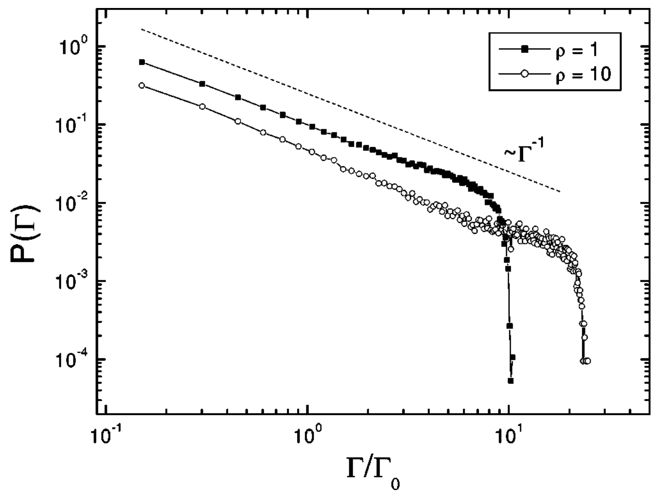

FIG. 2. $P(\Gamma)$ calculated for up to 50 configurations of 2500 scatterers randomly distributed in a square for $\rho=1$ (full squares) and $\rho=10$ (open circles) scatterers per wavelength squared. The normalization of Γ, as well as the significance of the solid and dashed lines, is the same as in Fig. 1. The value of the Thouless frequency for resonant scatterers is $\Gamma_T/\Gamma_0 \simeq 0.002$ and $\Gamma_T/\Gamma_0 \simeq 0.0002$ for $\rho=1$ and $\rho=10$, respectively.

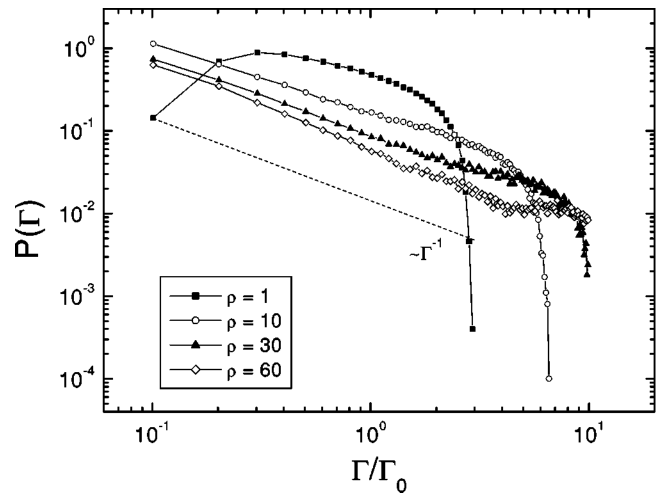

FIG. 3. $P(\Gamma)$ calculated for 100 configurations of 1000 scatterers randomly distributed in a sphere for $\rho=1$ (full squares), $\rho=10$ (open circles), $\rho=30$ (full triangles), and $\rho=60$ (open diamonds) scatterers per wavelength cubed. The normalization of Γ, as well as the significance of the solid and dashed lines, is the same as in Fig. 1.

principle all eigenstates are exponentially localized but the localization length ξ may be macroscopically large for low disorder according to [23]

$$\xi \simeq \ell \exp(\pi k_e \ell /2), \tag{3}$$

with k_e the effective wave number, which takes into account renormalized diffusion. Localization is expected to occur when ξ is smaller than the system size R. The Γ^{-1} decay of $P(\Gamma)$ in Fig. 2 is clearly visible for both values of ρ used, with an exponent very close to -1, in excellent agreement with Eq. (1). Notice that the range of the power law broadens as ρ increases. Increasing ρ means decreasing ℓ and, according to Eq. (3), a rapidly decreasing ξ. The range of the algebraic decay $P(\Gamma)\sim\Gamma^{-1}$ is expected to be broader as more and more states become localized. Such a behavior was also reported in numerical calculations within the Anderson model [10]. For large Γ, $P(\Gamma)$ decays again faster than algebraically as in the 1D case. To confirm that the system is indeed in the localized regime, let us estimate the ratio ξ/R from Eq. (3). Since $\ell/R=\pi/(2\sqrt{N\rho})$, the system can be said to be localized ($\xi/R<1$) when $k_e\ell \approx 2$ for $N=2500$ and for both values of ρ ($\rho=1$ and $\rho=10$) used in Fig. 2. This value of $k_e\ell$ is not too far from the "bare" estimates for the value of $k\ell=\pi^2/\rho$ in Fig. 2: $k\ell\approx 10$ for $\rho=1$ and $k\ell\approx 1$ for $\rho=10$, where we use the vacuum wave number $k=2\pi/\lambda$ and not the effective wave number k_e. We conclude that the localized scenario is valid. Consequently, the Thouless frequency Γ_T is not the appropriate characteristic decay rate in this case, as in 1D. For resonant scatterers in 2D, the ratio of Γ_T to Γ_0 is $\Gamma_T/\Gamma_0 \sim 2(\ell/R)^2=\pi^2/2N\rho$. For the values used in Fig. 2, we have $\Gamma_T/\Gamma_0\simeq 0.002$ and $\Gamma_T/\Gamma_0 \simeq 0.0002$, corresponding to $\rho=1$ and $\rho=10$, respectively.

In Fig. 3 the 3D case is considered, where $P(\Gamma)$ is calculated for systems composed by 1000 point scatterers randomly distributed in a sphere (radius R) for $\rho=1$, $\rho=10$, $\rho=30$, and $\rho=60$ scatterers per wavelength cubed. In 3D, the system is expected to undergo, upon varying the degree of disorder, a transition from extended states to localized states. It is therefore interesting to investigate if and how this transition manifests itself in $P(\Gamma)$. As in the 2D case, we notice that, as ρ increases, the range of the algebraic decay $P(\Gamma)\sim\Gamma^{-\alpha}$ increases. We also remark that, as ρ increases, the *associated exponents tend more and more to the value* -1. The exponents, obtained by a linear fit in the range where the power law is present, are $\alpha\approx 0.76$ for $\rho=1$, $\alpha\approx 0.83$ for $\rho=10$, $\alpha\approx 0.95$ for $\rho=30$, and $\alpha\approx 1.1$ for $\rho=60$. This suggests, according to Eq. (1), the onset of the localized regime for higher ρ. In fact, the Ioffe-Regel criterion for localization ($k\ell<1$) is estimated to be satisfied for $\rho>2\pi^2\approx 20$ for scatterers at resonance. This condition is fulfilled for $\rho=30$ and $\rho=60$, for which α is very close to 1, showing that the systems with these densities are indeed in the localized regime and confirming that the power law $P(\Gamma)\sim\Gamma^{-1}$ can be considered a genuine signature of Anderson localization of light. We anticipate that for $N,R\to\infty$ at constant ρ, the transition from the localized regime ($\alpha=1$) to the diffusive regime will become even more evident. Once again, note that $P(\Gamma)$ decays faster than a power law for very large Γ.

In 3D, the Thouless frequency Γ_T is the real characteristic internal decay rate of system since here there is a real diffusive regime, in contrast to the 1D and 2D cases. For this reason, we exhibit in Fig. 4 $P(\Gamma)$ for the same optical densities of Fig. 3 but now with Γ normalized to the Thouless frequency Γ_T, with Γ_T given by $\Gamma_T/\Gamma_0=2(\ell/R)^2 =2(4\pi/3)^{2/3}[\pi/(N^{1/3}\rho^{2/3})]^2$. For low ρ ($\rho=1$ and $\rho=10$), $P(\Gamma)$ is peaked near Γ_T, showing that the system is in the diffusive regime. Notice that there is a nonvanishing probability to find modes that live much longer than t_T even in

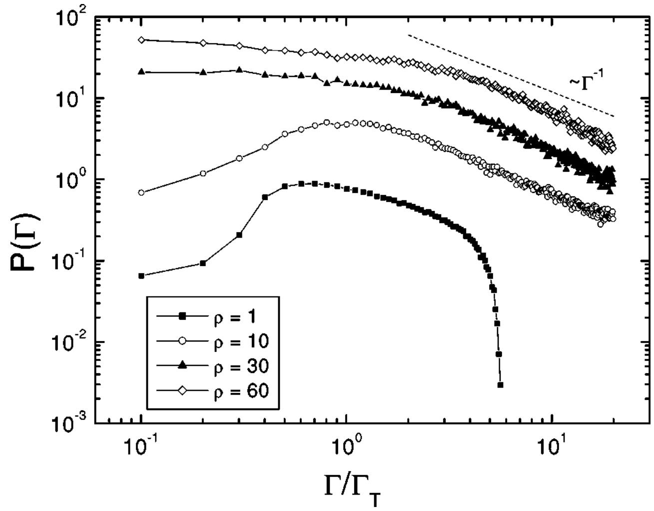

FIG. 4. $P(\Gamma)$ as in Fig. 3, but now Γ is normalized by the Thouless frequency Γ_T.

the diffusive regime, the so-called "prelocalized" modes [24]. As ρ increases, we observe that $P(\Gamma)$ is no longer centered at Γ_T and that the probability to find a mode with resonance width smaller than Γ_T also increases. This means that, on average, the modes live longer than t_T. At the same time, Fig. 3 shows that localization manifests itself in $P(\Gamma)$ not only via the broadening of the power law range but also via the fact that the associated exponents approach to -1. We conclude again that the $P(\Gamma)\sim\Gamma^{-1}$ behavior is an unambiguous signature of Anderson localization of light in open media. It must be mentioned that the present 3D study may be relevant for recent multiple light scattering experiments in atomic media [25], for which modeling the scatterers by pointlike dipoles constitutes an excellent approximation, though with a varying density ρ.

In summary, we have studied the distribution of resonance widths $P(\Gamma)$ in 1D, 2D, and 3D multiple light scattering systems composed of randomly distributed pointlike scalar dipoles. We have developed a simple physical argument, based on the exponential decay of localized eigenfunctions, to show that $P(\Gamma)$ should follow a universal power law $P(\Gamma)\sim\Gamma^{-1}$ decay in the localized regime. This prediction was confirmed by extensive numerical calculations and demonstrates that the $P(\Gamma)\sim\Gamma^{-1}$ behavior can be interpreted as an unambiguous signature of Anderson localization of light in open media.

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[1] For a recent overview, see, for instance, *Wave Scattering in Complex Media, from Theory to Applications*, edited by S. E. Skipetrov and B. A. van Tiggelen (Kluwer, Dordrecht, 2003); B. A. van Tiggelen, in *Wave Diffusion in Complex Media*, edited by J.-P. Fouque (Kluwer, Dordrecht, 1999).
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